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LETTER TO THE EDITOR

Continuity of quantum conditional information

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Abstract
We prove continuity of quantum conditional information $S(\rho^{12}|\rho^2)$ with respect to the uniform convergence of states and obtain a bound which is independent of the dimension of the second party. This can, e.g., be used to prove the continuity of squashed entanglement.

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A, generally mixed, state of a bipartite system is given by a density matrix ρ^{12} on a Hilbert space $\mathfrak{H}^{12} = \mathfrak{H}^1 \otimes \mathfrak{H}^2$. We shall, in order to avoid technical complications, restrict our attention to finite dimensional systems and not distinguish between the density matrix ρ^{12} and its associated expectation functional

$$a \mapsto \rho^{12}(a) := \mathrm{Tr}\,\rho^{12}a \qquad a \text{ a linear operator on } \mathfrak{H}^{12}.$$

The restrictions of ρ^{12} to the subsystems 1 and 2 are denoted by ρ^1 and ρ^2, e.g.

$$\rho^1(a) := \rho^1(a \otimes \mathbf{1}) = \mathrm{Tr}\,\rho^{12} a \otimes \mathbf{1} \qquad a \text{ a linear operator on } \mathfrak{H}^1.$$

The von Neumann entropy $S(\rho)$ of a state ρ is the quantity $\mathrm{Tr}\,\eta(\rho)$ with $\eta(x) := -x \log x$ for $0 < x \leqslant 1$ and $\eta(0) = 0$. The conditional information $S(\rho^{12}|\rho^2)$ of ρ^{12} with respect to the second system is the quantity

$$S(\rho^{12}|\rho^2) := S(\rho^{12}) - S(\rho^2)$$

$S(\rho^{12}|\rho^2)$ is also called conditional entropy. Finally, we need the uniform distance between states:

$$\|\rho - \sigma\|_1 := \sup_{a, \|a\| \leqslant 1} |\rho(a) - \sigma(a)| = \mathrm{Tr}\,|\rho - \sigma|.$$

In this last equation $|a|$ denotes the absolute value of a matrix (or an operator on Hilbert space). It is given by $|a| := \sqrt{a^* a}$. For the matrix case, the eigenvalues of $|a|$ are often called the singular values of a.

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Theorem. *Take any two states ρ^{12} and σ^{12} on $\mathfrak{H}^{12} = \mathfrak{H}^1 \otimes \mathfrak{H}^2$ such that $\epsilon := \|\rho^{12} - \sigma^{12}\| < 1$ and let d_1 be the dimension of $\mathfrak{H}^1$, then the following estimate holds:*

$$\left|S(\rho^{12}|\rho^2) - S(\sigma^{12}|\sigma^2)\right| \leqslant 4\epsilon \log d_1 + 2\eta(1-\epsilon) + 2\eta(\epsilon). \tag{1}$$

In particular, the right-hand side of (1) does not explicitly depend on the dimension of $\mathfrak{H}^2$.

Proof. The basic step in the proof is the introduction of an auxiliary state

$$\gamma^{12} := (1-\epsilon)\rho^{12} + |\rho^{12} - \sigma^{12}|.$$

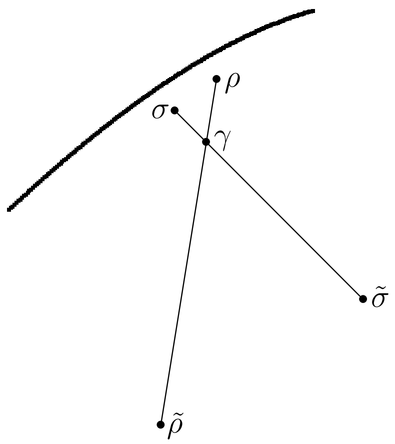
As $0 \leqslant \epsilon \leqslant 1$, γ^{12} is indeed a state. Next, dealing only with the non-trivial case $\epsilon > 0$, we introduce two new states

$$\tilde{\rho}^{12} := \frac{1}{\epsilon}|\rho^{12} - \sigma^{12}| \qquad \text{and} \qquad \tilde{\sigma}^{12} := \frac{1-\epsilon}{\epsilon}(\rho^{12} - \sigma^{12}) + \frac{1}{\epsilon}|\rho^{12} - \sigma^{12}|.$$

A direct computation shows that

$$\gamma^{12} = (1-\epsilon)\rho^{12} + \epsilon\tilde{\rho}^{12} = (1-\epsilon)\sigma^{12} + \epsilon\tilde{\sigma}^{12}.$$

The situation precisely matches that of the theorem of Thales of Milete in planar geometry.



We now estimate

$$\begin{aligned}|S(\rho^{12}|\rho^2) - S(\sigma^{12}|\sigma^2)| &\leqslant |S(\rho^{12}|\rho^2) - S(\gamma^{12}|\gamma^2)| + |S(\sigma^{12}|\sigma^2) - S(\gamma^{12}|\gamma^2)| \\ &\leqslant 4\epsilon \log d_1 + 2\eta(1-\epsilon) + 2\eta(\epsilon).\end{aligned}$$

The last inequality follows from the lemma. $\square$

Lemma. *Let ρ^{12} and $\tilde{\rho}^{12}$ be two states on $\mathfrak{H}^{12}$, let $0 \leqslant \epsilon \leqslant 1$ and put $\gamma^{12} := (1-\epsilon)\rho^{12} + \epsilon\tilde{\rho}^{12}$. If d_1 is the dimension of $\mathfrak{H}_1$, then*

$$|S(\rho^{12}|\rho^2) - S(\gamma^{12}|\gamma^2)| \leqslant 2\epsilon \log d_1 + \eta(1-\epsilon) + \eta(\epsilon).$$

Proof. As the conditional entropy is concave, see [3], we have

$$S(\gamma^{12}|\gamma^2) \geqslant (1-\epsilon)S(\rho^{12}|\rho^2) + \epsilon S(\tilde{\rho}^{12}|\tilde{\rho}^2).$$

Therefore

$$\begin{aligned}S(\rho^{12}|\rho^2) - S(\gamma^{12}|\gamma^2) &\leqslant \epsilon(S(\rho^{12}|\rho^2) - S(\tilde{\rho}^{12}|\tilde{\rho}^2)) \\ &\leqslant 2\epsilon \log d_1.\end{aligned} \tag{2}$$

Next, we use the concavity of the entropy

$$S(\gamma^2) \geqslant (1-\epsilon)S(\rho^2) + \epsilon S(\tilde{\rho}^2)$$

and the upper bound

$$S(\gamma^{12}) \leqslant (1-\epsilon)S(\rho^{12}) + \epsilon S(\tilde{\rho}^{12}) + \eta(1-\epsilon) + \eta(\epsilon)$$

to obtain

$$\begin{aligned} S(\gamma^{12}|\gamma^2) &= S(\gamma^{12}) - S(\gamma^2) \\ &\leqslant (1-\epsilon)(S(\rho^{12}) - S(\rho^2)) + \epsilon(S(\tilde{\rho}^{12}) - S(\tilde{\rho}^2)) + \eta(1-\epsilon) + \eta(\epsilon). \end{aligned}$$

Therefore

$$\begin{aligned} S(\rho^{12}|\rho^2) - S(\gamma^{12}|\gamma^2) &\geqslant \epsilon(S(\rho^{12}|\rho^2) - S(\tilde{\rho}^{12}|\tilde{\rho}^2)) - \eta(1-\epsilon) - \eta(\epsilon) \\ &\geqslant -2\epsilon \log d_1 - \eta(1-\epsilon) - \eta(\epsilon). \end{aligned} \tag{3}$$

Combining (2) and (3), the lemma follows. □

M Horodecki [2] kindly explained us the relevance of continuity of conditional quantum information for obtaining the asymptotic continuity of the newly introduced squashed entanglement for mixed bipartite states. This quantity is the smallest conditional mutual information computed over the set of all finite dimensional extensions of the state:

$$E_{\mathrm{sq}(\rho^{12})} := \inf_{\rho^{123}} \tfrac{1}{2}(S(\rho^{13}|\rho^3) - S(\rho^{123}|\rho^{23})).$$

It was introduced in [4] and shown to vanish on separable states and to be additive, see [1]. Continuity in the state, however, remained an open issue, the missing piece being precisely our theorem.

As conditional entropy is a rather basic quantity in quantum information theory, its continuity might prove useful in another context. We would also like to point out that the theorem provides a concise proof, though with non-optimal constants, of the continuity of von Neumann entropy

$$|S(\rho) - S(\sigma)| \leqslant 2\epsilon \log d + 2\eta(\epsilon) + 2\eta(1-\epsilon)$$

with $\epsilon := \|\rho - \sigma\|_1$ and d the dimension of the space spanned by ρ and σ.

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